An Introduction to Lagrangian Mechanics

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1 What is Lagrangian mechanics?

Lagrangian mechanics is a formulation of classical mechanics built around a scalar function

$$L(q, \dot{q}, t) = T(q, \dot{q}, t) - V(q, t),$$

where

- $q = (q_1, \dots, q_n)$ are *generalized coordinates*,
- $\dot{q} = (\dot{q}_1, \dots, \dot{q}_n)$ are generalized velocities,
- T is kinetic energy,
- V is potential energy.

Its main advantage is that it handles constraints and non-Cartesian coordinates naturally. For many problems, especially pendulums, rigid bodies, and robot manipulators, Newton's law written in Cartesian coordinates is awkward, while the Lagrangian method is systematic.
A good mental model is this:

> Choose coordinates that describe the configuration; write down the total kinetic and potential energies; then apply the Euler–Lagrange equations.

2 Configuration space and generalized coordinates

A mechanical system does not need to be described by Cartesian coordinates of every particle. Instead, we use a minimal set of variables that determine the configuration.

2.1 Examples

- A particle in the plane: $q = (x, y)$.
- A pendulum of fixed length ℓ: one angle $q = \theta$.
- A planar 2-link robot arm: joint angles $q = (q_1, q_2)$.

The set of all possible configurations is called the *configuration space*. Locally, one may view it as an n-dimensional manifold with coordinates $q_1, \dots, q_n$.

3 The principle of stationary action

Given a path $q(t)$ on $[t_1, t_2]$, define the *action*

$$S[q] = \int_{t_1}^{t_2} L(q(t), \dot{q}(t), t)\, \mathrm{d}t.$$

The physical trajectory is characterized by the condition that $S[q]$ is stationary under small variations $\delta q(t)$ with fixed endpoints:

$$\delta q(t_1) = \delta q(t_2) = 0.$$

This yields the Euler–Lagrange equations.

3.1 Derivation of the Euler–Lagrange equations

Let $q_\varepsilon(t) = q(t) + \varepsilon\eta(t)$, where $\eta(t_1) = \eta(t_2) = 0$. Then

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} S[q_\varepsilon]\bigg|_{\varepsilon=0} = \int_{t_1}^{t_2} \sum_{i=1}^{n} \left(\frac{\partial L}{\partial q_i}\eta_i + \frac{\partial L}{\partial \dot{q}_i}\dot{\eta}_i \right) \mathrm{d}t.$$

Integrating the second term by parts,

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_i}\dot{\eta}_i \, \mathrm{d}t = \left[\frac{\partial L}{\partial \dot{q}_i}\eta_i\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\eta_i \, \mathrm{d}t.$$

The boundary term vanishes because $\eta_i(t_1) = \eta_i(t_2) = 0$. Hence

$$\delta S = \int_{t_1}^{t_2} \sum_{i=1}^{n} \left(\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i}\right)\eta_i \, \mathrm{d}t.$$

Since the η_i are arbitrary, we obtain

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \qquad i = 1, \ldots, n.}$$

These are the Euler–Lagrange equations.

4 How to build the Lagrangian

For many systems,

$$L = T - V.$$

The practical recipe is:

(a) Choose generalized coordinates q.

(b) Express the position of each mass in terms of q.

(c) Differentiate to get velocities, hence kinetic energy T.

(d) Write the potential energy V.

(e) Compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0.$$

4.1 Kinetic energy in quadratic form

For many finite-dimensional mechanical systems,

$$T(q, \dot{q}) = \frac{1}{2}\dot{q}^{\mathrm{T}} M(q)\dot{q},$$

where $M(q)$ is the *mass matrix*, symmetric and typically positive definite. Then

$$L(q, \dot{q}) = \frac{1}{2}\dot{q}^{\mathrm{T}} M(q)\dot{q} - V(q).$$

This form is extremely common in robotics.

5 First examples

5.1 Free particle in $\mathbb{R}^n$

Take $q = x \in \mathbb{R}^n$. The kinetic energy is

$$T = \frac{1}{2}m\dot{x}^{\mathrm{T}}\dot{x}, \qquad V = 0.$$

So

$$L = \frac{1}{2} m \dot{x}^{\mathrm{T}} \dot{x}.$$

The Euler–Lagrange equations give

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x}) = 0 \quad \Longrightarrow \quad m\ddot{x} = 0.$$

Thus the velocity is constant.

5.2 Particle in a potential

If

$$L = \frac{1}{2} m \dot{x}^{\mathrm{T}} \dot{x} - V(x),$$

then

$$m\ddot{x} + \nabla V(x) = 0,$$

or equivalently

$$m\ddot{x} = -\nabla V(x).$$

This recovers Newton's second law with conservative force $F = -\nabla V$.

6 Classic example: the simple pendulum

A pendulum of mass m and length ℓ moves in a vertical plane. Use the angle θ from the downward vertical.

6.1 Geometry

Its position is

$$x = \ell \sin\theta, \qquad y = -\ell \cos\theta.$$

Hence

$$\dot{x} = \ell \cos\theta \, \dot{\theta}, \qquad \dot{y} = \ell \sin\theta \, \dot{\theta},$$

so

$$\dot{x}^2 + \dot{y}^2 = \ell^2 \dot{\theta}^2.$$

6.2 Lagrangian

The kinetic energy is

$$T = \frac{1}{2} m \ell^2 \dot{\theta}^2.$$

The potential energy may be taken as

$$V = mg\ell(1 - \cos\theta).$$

Thus

$$L = \frac{1}{2} m \ell^2 \dot{\theta}^2 - mg\ell(1 - \cos\theta).$$

6.3 Equation of motion

Compute

$$\frac{\partial L}{\partial \dot{\theta}} = m\ell^2\dot{\theta}, \qquad \frac{\partial L}{\partial \theta} = -mg\ell\sin\theta.$$

Therefore

$$m\ell^2\ddot{\theta} + mg\ell\sin\theta = 0.$$

After dividing by $m\ell$,

$$\boxed{\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0.}$$

For small angles, $\sin\theta \approx \theta$, giving the linear approximation

$$\ddot{\theta} + \frac{g}{\ell}\theta = 0,$$

which is simple harmonic motion.

7 Classic example: a particle in polar coordinates

Consider a particle of mass m moving in the plane. In polar coordinates (r, ϕ),

$$x = r\cos\phi, \qquad y = r\sin\phi.$$

Then

$$\dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\phi}^2,$$

so

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2).$$

If the potential is central, $V = V(r)$, then

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - V(r).$$

The Euler–Lagrange equations become

$$m\ddot{r} - mr\dot{\phi}^2 + V'(r) = 0,$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}(mr^2\dot{\phi}) = 0.$$

Thus

$$mr^2\dot{\phi} = \text{constant}.$$

This is angular momentum conservation. The angle ϕ is a *cyclic coordinate*: L does not depend explicitly on ϕ.

8 Conserved quantities

8.1 Generalized momentum

For each coordinate q_i, define the conjugate momentum

$$p_i = \frac{\partial L}{\partial \dot{q}_i}.$$

If L does not depend on q_i explicitly, then

$$\frac{\partial L}{\partial q_i} = 0 \quad \Longrightarrow \quad \dot{p}_i = 0.$$

So p_i is conserved.
Such a coordinate is called *cyclic* or *ignorable*.

8.2 Energy

Define the energy function

$$E(q, \dot{q}, t) = \sum_{i=1}^{n} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L.$$

If L has no explicit time dependence, then E is conserved.
For a standard mechanical system with

$$L = T - V,$$

and where T is quadratic in velocities, one gets

$$E = T + V.$$

9 Classic example: the Atwood machine

Consider two masses m_1, m_2 connected by a massless string over an ideal pulley. Let x be the downward displacement of m_1. Because the string length is fixed, the other mass moves by $-x$.

9.1 Lagrangian

Both masses have speed $|\dot{x}|$. Hence

$$T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{x}^2 = \frac{1}{2}(m_1 + m_2)\dot{x}^2.$$

If downward is positive for m_1, then a convenient potential is

$$V = -m_1 g x + m_2 g x = (m_2 - m_1) g x.$$

So

$$L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 - (m_2 - m_1) g x.$$

9.2 Equation of motion

The Euler–Lagrange equation gives

$$(m_1 + m_2)\ddot{x} - (m_1 - m_2) g = 0,$$

hence

$$\boxed{\ddot{x} = \frac{m_1 - m_2}{m_1 + m_2} g.}$$

The constraint was handled automatically by choosing one coordinate.

10 Why constraints are easier here

Suppose a particle is constrained to move on a circle of radius R. In Cartesian coordinates one must enforce

$$x^2 + y^2 = R^2.$$

In Lagrangian mechanics it is better to parameterize the constraint directly:

$$x = R\cos\theta, \qquad y = R\sin\theta.$$

Then the constraint disappears from the equations, and θ is the only variable.
This is a major conceptual advantage:

> Instead of writing forces of constraint explicitly, choose coordinates that already respect the constraint.

11 Small oscillations near equilibrium

Suppose q_* is an equilibrium point of a conservative system with

$$L = \frac{1}{2}\dot{q}^{\mathrm{T}} M(q)\dot{q} - V(q).$$

At equilibrium,

$$\nabla V(q_*) = 0.$$

For small perturbations $\eta = q - q_*$, approximate

$$M(q) \approx M(q_*) = M_*,$$

and

$$V(q) \approx V(q_*) + \frac{1}{2}\eta^{\mathrm{T}} K\eta, \qquad K = \nabla^2 V(q_*).$$

Then

$$L_{\mathrm{lin}}(\eta, \dot{\eta}) = \frac{1}{2}\dot{\eta}^{\mathrm{T}} M_*\dot{\eta} - \frac{1}{2}\eta^{\mathrm{T}} K\eta,$$

and the linearized equations are

$$M_*\ddot{\eta} + K\eta = 0.$$

This is the starting point for normal modes and vibration analysis.

12 Robotics example: a single rotary joint under gravity

A rigid link of length ℓ and mass m rotates in a vertical plane about one end. Let q be the joint angle, measured from the downward vertical. Assume the center of mass is at distance c from the joint, and the moment of inertia about the center of mass is I_{cm}.

12.1 Kinetic and potential energy

The center of mass moves on a circle of radius c, so its translational kinetic energy is

$$T_{\mathrm{trans}} = \frac{1}{2}mc^2\dot{q}^2.$$

The rotational kinetic energy about the center of mass is

$$T_{\mathrm{rot}} = \frac{1}{2}I_{\mathrm{cm}}\dot{q}^2.$$

Hence

$$T = \frac{1}{2}(I_{\mathrm{cm}} + mc^2)\dot{q}^2.$$

The potential energy is

$$V = mgc(1 - \cos q).$$

Therefore

$$L = \frac{1}{2}(I_{\mathrm{cm}} + mc^2)\dot{q}^2 - mgc(1 - \cos q).$$

12.2 Equation of motion

The Euler–Lagrange equation gives

$$(I_{\mathrm{cm}} + mc^2)\ddot{q} + mgc\sin q = 0.$$

This is the rigid-link analogue of the pendulum equation.
In robotics, this is the dynamics of one revolute joint with gravity.

13 Robotics example: a planar 2-link manipulator

This is a standard model in robot dynamics.

13.1 Setup

Consider two planar revolute joints with angles

$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}.$$

Let

- ℓ_1, ℓ_2 be link lengths,
- c_1, c_2 be center-of-mass distances from the upstream joints,
- m_1, m_2 be masses,
- I_1, I_2 be planar moments of inertia about the centers of mass.

The first center of mass is at

$$p_1 = \begin{bmatrix} c_1 \cos q_1 \\ c_1 \sin q_1 \end{bmatrix},$$

and the second at

$$p_2 = \begin{bmatrix} \ell_1 \cos q_1 + c_2 \cos(q_1 + q_2) \\ \ell_1 \sin q_1 + c_2 \sin(q_1 + q_2) \end{bmatrix}.$$

13.2 Velocities

Differentiating,

$$\dot{p}_1 = \begin{bmatrix} -c_1 \sin q_1 \, \dot{q}_1 \\ c_1 \cos q_1 \, \dot{q}_1 \end{bmatrix}, \qquad \|\dot{p}_1\|^2 = c_1^2 \dot{q}_1^2.$$

For the second link,

$$\|\dot{p}_2\|^2 = \ell_1^2 \dot{q}_1^2 + c_2^2 (\dot{q}_1 + \dot{q}_2)^2 + 2\ell_1 c_2 \cos q_2 \, \dot{q}_1 (\dot{q}_1 + \dot{q}_2).$$

13.3 Energy

Hence

$$T = \frac{1}{2} m_1 c_1^2 \dot{q}_1^2 + \frac{1}{2} I_1 \dot{q}_1^2 + \frac{1}{2} m_2 \|\dot{p}_2\|^2 + \frac{1}{2} I_2 (\dot{q}_1 + \dot{q}_2)^2.$$

After collecting terms, this can be written as

$$T = \frac{1}{2} \dot{q}^{\mathrm{T}} M(q) \dot{q},$$

where

$$M(q) = \begin{bmatrix} a + 2b \cos q_2 & d + b \cos q_2 \\ d + b \cos q_2 & d \end{bmatrix},$$

with

$$a = I_1 + I_2 + m_1 c_1^2 + m_2 (\ell_1^2 + c_2^2), \qquad b = m_2 \ell_1 c_2, \qquad d = I_2 + m_2 c_2^2.$$

If the vertical coordinate is the second component, then the potential energy is

$$V = m_1 g c_1 \sin q_1 + m_2 g \big(\ell_1 \sin q_1 + c_2 \sin(q_1 + q_2)\big).$$

Thus

$$L(q, \dot{q}) = \frac{1}{2} \dot{q}^{\mathrm{T}} M(q) \dot{q} - V(q).$$

13.4 Equation structure

Carrying out the Euler–Lagrange equations leads to the standard robot form

$$\boxed{M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau,}$$

where

- $M(q)$ is the mass matrix,
- $C(q,\dot{q})\dot{q}$ contains Coriolis and centrifugal terms,
- $g(q) = \nabla_q V(q)$ is the gravity vector,
- τ is the vector of actuator torques.

For the passive case, $\tau = 0$.
One convenient expression is

$$g(q) = \begin{bmatrix} (m_1 c_1 + m_2 \ell_1) g \cos q_1 + m_2 c_2 g \cos(q_1 + q_2) \\ m_2 c_2 g \cos(q_1 + q_2) \end{bmatrix}.$$

A corresponding choice for the velocity-dependent terms is

$$C(q,\dot{q})\dot{q} = \begin{bmatrix} -b \sin q_2 \,(2\dot{q}_1 \dot{q}_2 + \dot{q}_2^2) \\ b \sin q_2 \, \dot{q}_1^2 \end{bmatrix}.$$

This example is important because it is essentially the prototype for rigid-body robot dynamics.

14 Robotics example: cart–pole

The cart–pole is a canonical control and robotics system.

14.1 Coordinates

Let

$$q = \begin{bmatrix} x \\ \theta \end{bmatrix},$$

where x is the cart position and θ is the pendulum angle from the upward vertical. Let the cart mass be M, the pole bob mass be m, and the pole length be ℓ.
The bob position is

$$p = \begin{bmatrix} x + \ell \sin\theta \\ \ell \cos\theta \end{bmatrix}.$$

Hence

$$\dot{p} = \begin{bmatrix} \dot{x} + \ell \cos\theta \, \dot{\theta} \\ -\ell \sin\theta \, \dot{\theta} \end{bmatrix},$$

so

$$\|\dot{p}\|^2 = \dot{x}^2 + 2\ell \cos\theta \, \dot{x}\dot{\theta} + \ell^2 \dot{\theta}^2.$$

14.2 Lagrangian

Thus

$$T = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m \|\dot{p}\|^2 = \frac{1}{2}(M+m)\dot{x}^2 + m\ell \cos\theta \, \dot{x}\dot{\theta} + \frac{1}{2} m \ell^2 \dot{\theta}^2.$$

The potential energy is

$$V = mg\ell \cos\theta.$$

So

$$L = \frac{1}{2}(M+m)\dot{x}^2 + m\ell \cos\theta \, \dot{x}\dot{\theta} + \frac{1}{2} m \ell^2 \dot{\theta}^2 - mg\ell \cos\theta.$$

14.3 Equations

The Euler–Lagrange equations yield

$$(M+m)\ddot{x} + m\ell(-\sin\theta\,\dot{\theta}^2 + \cos\theta\,\ddot{\theta}) = u,$$

and

$$m\ell^2\ddot{\theta} + m\ell\cos\theta\,\ddot{x} - mg\ell\sin\theta = 0,$$

where u is the horizontal control force on the cart.
This example shows how Lagrangian mechanics naturally incorporates actuation.

15 The robot form in general

For many mechanical systems used in robotics,

$$L(q,\dot{q}) = \frac{1}{2}\dot{q}^{\mathrm{T}} M(q)\dot{q} - V(q).$$

Then the equations can be organized as

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau.$$

This form is fundamental because:

- it separates inertia, velocity coupling, and gravity,
- it is coordinate-based and systematic,
- it is directly used in robot control, estimation, and simulation.

Some structural facts are worth remembering:

- $M(q)$ is symmetric,
- under mild assumptions, $M(q)$ is positive definite,
- $g(q) = \nabla_q V(q)$,
- velocity-dependent terms arise from derivatives of $M(q)$.

16 Common mistakes

(a) **Using too many coordinates.** Prefer minimal generalized coordinates.
(b) **Forgetting that T depends on velocities through the chain rule.**
(c) **Sign mistakes in potential energy.** Fix a reference and keep it consistent.
(d) **Confusing $\partial L/\partial q_i$ with $\mathrm{d}(\partial L/\partial \dot{q}_i)/\mathrm{d}t$.**
(e) **Linearizing too early.** Derive the exact equation first, then approximate if needed.

17 A compact derivation checklist

When solving a new problem, use the following template.

(a) Pick generalized coordinates q.
(b) Write all positions as functions of q.
(c) Differentiate to obtain velocities.

(d) Compute

$$T = \sum \frac{1}{2} m_i v_i^2 + \sum \frac{1}{2} \omega_i^{\mathrm{T}} I_i \omega_i.$$

(e) Compute V.

(f) Form $L = T - V$.

(g) Apply

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i^{\mathrm{nc}},$$

if non-conservative generalized forces Q_i^{nc} are present.

Here Q_i^{nc} can represent input torques, friction models, or external forcing. In robotics, one often writes them collectively as τ_i.

18 Brief note on Noether's principle

A full treatment belongs to advanced mechanics, but one simple slogan is important:

> Continuous symmetry implies conservation law.

Examples:

- Invariance under time translation $\Rightarrow$ energy conservation.
- Invariance under spatial translation $\Rightarrow$ linear momentum conservation.
- Invariance under rotation $\Rightarrow$ angular momentum conservation.

For beginner use, the practical version is:

- if L does not depend on a coordinate, the associated generalized momentum is conserved;
- if L does not depend explicitly on time, energy is conserved.

19 Where to go next

A natural progression is:

(a) constrained systems with Lagrange multipliers,

(b) Hamiltonian mechanics,

(c) rigid-body dynamics on SO(3) and SE(3),

(d) geometric mechanics and analytical mechanics for robotics.

For robotics in particular, the next important topic is how Lagrangian dynamics leads to the manipulator equation

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = \tau,$$

and how this connects to computed-torque control, inverse dynamics, and trajectory optimization.

Appendix: a few useful formulas

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0,& \\
p_i = \frac{\partial L}{\partial \dot{q}_i},& \\
E = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L,& \\
T = \frac{1}{2}\dot{q}^{\mathrm{T}} M(q)\dot{q},& \\
L = T - V.&
\end{aligned}
$$