Theorem (Carathéodory Extension Theorem)**.** *Let X be a set, let $\mathcal{A}$ be an algebra of subsets of X, and let $\mu_0\colon \mathcal{A} \to [0,\infty]$ be a premeasure. Then there exists a measure μ on $\sigma(\mathcal{A})$ such that*

$$\mu(A) = \mu_0(A), \qquad A \in \mathcal{A}.$$

If μ_0 is σ-finite, then this extension is unique.

1 Construction of the Outer Measure

For $E \subseteq X$, define

$$\mu^*(E) = \inf \left\{ \sum_{n=1}^{\infty} \mu_0(A_n) : A_n \in \mathcal{A},\ E \subseteq \bigcup_{n=1}^{\infty} A_n \right\}.$$

We first show that μ^* is an outer measure. Clearly $\mu^*(\varnothing) = 0$. If $E \subseteq F$, every countable $\mathcal{A}$-cover of F is also a cover of E, so $\mu^*(E) \le \mu^*(F)$.

It remains to prove countable subadditivity. Let $E = \bigcup_{k=1}^{\infty} E_k$. If $\sum_{k=1}^{\infty} \mu^*(E_k) = \infty$, there is nothing to prove. Otherwise, fix $\varepsilon > 0$. For each k, choose $A_{k,j} \in \mathcal{A}$ such that

$$E_k \subseteq \bigcup_{j=1}^{\infty} A_{k,j}, \qquad \sum_{j=1}^{\infty} \mu_0(A_{k,j}) \le \mu^*(E_k) + \frac{\varepsilon}{2^k}.$$

Then $\{A_{k,j}\}_{k,j \ge 1}$ is a countable cover of E, and hence

$$\mu^*(E) \le \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \mu_0(A_{k,j}) \le \sum_{k=1}^{\infty} \mu^*(E_k) + \varepsilon.$$

Letting $\varepsilon \downarrow 0$ gives

$$\mu^*\left(\bigcup_{k=1}^{\infty} E_k \right) \le \sum_{k=1}^{\infty} \mu^*(E_k).$$

Thus μ^* is an outer measure.

2 Carathéodory Measurable Sets

A set $E \subseteq X$ is called Carathéodory measurable if

$$\mu^*(T) = \mu^*(T \cap E) + \mu^*(T \setminus E) \qquad \text{for every } T \subseteq X.$$

Let $\mathcal{M}$ denote the collection of all Carathéodory measurable sets. We show that $\mathcal{M}$ is a σ-algebra and that $\mu^*|_{\mathcal{M}}$ is a measure.

2.1 The class $\mathcal{M}$ is a σ-algebra

Clearly $\varnothing \in \mathcal{M}$, and the defining equality is symmetric in E and E^c, so $E \in \mathcal{M}$ implies $E^c \in \mathcal{M}$.

Suppose $E, F \in \mathcal{M}$. For every $T \subseteq X$,

$$\begin{aligned}
\mu^*(T) &= \mu^*(T \cap E) + \mu^*(T \setminus E) \\
&= \mu^*(T \cap E) + \mu^*((T \setminus E) \cap F) + \mu^*(T \setminus (E \cup F)) \\
&\ge \mu^*(T \cap (E \cup F)) + \mu^*(T \setminus (E \cup F)).
\end{aligned}$$

The reverse inequality follows from subadditivity, so $E \cup F \in \mathcal{M}$. Hence $\mathcal{M}$ is an algebra.

Now let $E_1, E_2, \ldots \in \mathcal{M}$ be pairwise disjoint and set $F_n = \bigcup_{k=1}^n E_k$. Repeated application of the Carathéodory condition gives, for every $T \subseteq X$,

$$\mu^*(T) = \sum_{k=1}^n \mu^*(T \cap E_k) + \mu^*(T \setminus F_n).$$

Let $E = \bigcup_{k=1}^\infty E_k$. Since $T \setminus F_n \supseteq T \setminus E$,

$$\mu^*(T) \geq \sum_{k=1}^n \mu^*(T \cap E_k) + \mu^*(T \setminus E).$$

Letting $n \to \infty$ and using subadditivity,

$$\begin{aligned} \mu^*(T) &\geq \sum_{k=1}^\infty \mu^*(T \cap E_k) + \mu^*(T \setminus E) \\ &\geq \mu^*(T \cap E) + \mu^*(T \setminus E). \end{aligned}$$

Again the reverse inequality follows from subadditivity, so $E \in \mathcal{M}$. Any countable union can be disjointified, and therefore $\mathcal{M}$ is a σ-algebra.

2.2 The restriction $\mu^*|_{\mathcal{M}}$ is a measure

Let $E_1, E_2, \ldots \in \mathcal{M}$ be pairwise disjoint and put $E = \bigcup_{k=1}^\infty E_k$. Taking $T = E$ in the preceding argument gives

$$\mu^*(E) \geq \sum_{k=1}^\infty \mu^*(E_k).$$

Countable subadditivity gives the reverse inequality. Therefore

$$\mu^*\left(\bigcup_{k=1}^\infty E_k\right) = \sum_{k=1}^\infty \mu^*(E_k),$$

so $\mu^*|_{\mathcal{M}}$ is a measure.

3 Measurability of the Original Algebra

We prove that $\mathcal{A} \subseteq \mathcal{M}$. Fix $A \in \mathcal{A}$ and $T \subseteq X$. Let $T \subseteq \bigcup_{n=1}^\infty A_n$, where $A_n \in \mathcal{A}$. Since $\mathcal{A}$ is an algebra, $A_n \cap A, A_n \setminus A \in \mathcal{A}$, and finite additivity of μ_0 gives

$$\mu_0(A_n) = \mu_0(A_n \cap A) + \mu_0(A_n \setminus A).$$

Moreover,

$$T \cap A \subseteq \bigcup_{n=1}^\infty (A_n \cap A), \qquad T \setminus A \subseteq \bigcup_{n=1}^\infty (A_n \setminus A).$$

Therefore

$$\begin{aligned} \mu^*(T \cap A) + \mu^*(T \setminus A) &\leq \sum_{n=1}^\infty \mu_0(A_n \cap A) + \sum_{n=1}^\infty \mu_0(A_n \setminus A) \\ &= \sum_{n=1}^\infty \mu_0(A_n). \end{aligned}$$

Taking the infimum over all countable $\mathcal{A}$-covers of T yields

$$\mu^*(T \cap A) + \mu^*(T \setminus A) \leq \mu^*(T).$$

The reverse inequality follows from subadditivity, so $A \in \mathcal{M}$. Hence

$$\mathcal{A} \subseteq \mathcal{M}, \qquad \sigma(\mathcal{A}) \subseteq \mathcal{M}.$$

4 Agreement with the Premeasure and Existence

Fix $A \in \mathcal{A}$. The one-set cover $A \subseteq A$ gives $\mu^*(A) \le \mu_0(A)$. Conversely, suppose $A \subseteq \bigcup_{n=1}^\infty A_n$ with $A_n \in \mathcal{A}$. Define

$$B_1 = A \cap A_1, \qquad B_n = A \cap \left(A_n \setminus \bigcup_{k=1}^{n-1} A_k \right), \quad n \ge 2.$$

Then $B_n \in \mathcal{A}$, the sets B_n are pairwise disjoint, $A = \bigcup_{n=1}^\infty B_n$, and $B_n \subseteq A_n$. Since μ_0 is a premeasure,

$$\mu_0(A) = \sum_{n=1}^\infty \mu_0(B_n) \le \sum_{n=1}^\infty \mu_0(A_n).$$

Taking the infimum over all such covers yields $\mu_0(A) \le \mu^*(A)$. Therefore

$$\mu^*(A) = \mu_0(A), \qquad A \in \mathcal{A}.$$

Since $\sigma(\mathcal{A}) \subseteq \mathcal{M}$, the restriction

$$\mu = \mu^*|_{\sigma(\mathcal{A})}$$

is a measure on $\sigma(\mathcal{A})$ extending μ_0. This proves existence.

5 Uniqueness under σ-Finiteness

Assume that μ_0 is σ-finite, and let μ and ν be two measures on $\sigma(\mathcal{A})$ extending μ_0. Choose $A_n \in \mathcal{A}$ such that $X = \bigcup_{n=1}^\infty A_n$ and $\mu_0(A_n) < \infty$. Replacing A_n by $\bigcup_{k=1}^n A_k$, we may assume that $A_n \uparrow X$ and still $\mu_0(A_n) < \infty$.

For each n, define

$$\mathcal{D}_n = \{E \in \sigma(\mathcal{A}) : \mu(E \cap A_n) = \nu(E \cap A_n)\}.$$

Because $\mu(A_n) = \nu(A_n) = \mu_0(A_n) < \infty$, $\mathcal{D}_n$ is a Dynkin system: it contains X, it is closed under complements by subtraction from $\mu(A_n) = \nu(A_n)$, and it is closed under countable disjoint unions by countable additivity. Moreover, if $A \in \mathcal{A}$, then $A \cap A_n \in \mathcal{A}$, so

$$\mu(A \cap A_n) = \mu_0(A \cap A_n) = \nu(A \cap A_n).$$

Thus $\mathcal{A} \subseteq \mathcal{D}_n$. Since $\mathcal{A}$ is a π-system, the π-λ theorem gives

$$\sigma(\mathcal{A}) \subseteq \mathcal{D}_n.$$

Hence, for every $E \in \sigma(\mathcal{A})$ and every n,

$$\mu(E \cap A_n) = \nu(E \cap A_n).$$

Since $E \cap A_n \uparrow E$, continuity from below yields

$$\mu(E) = \lim_{n\to\infty} \mu(E \cap A_n) = \lim_{n\to\infty} \nu(E \cap A_n) = \nu(E).$$

Therefore $\mu = \nu$ on $\sigma(\mathcal{A})$, proving uniqueness.