Path Regularity of Brownian Motion

Let $B = (B_t)_{t \geq 0}$ be a standard Brownian motion. We prove that, with probability one, its sample path is nowhere differentiable and has infinite total variation on every nontrivial interval.

1 Nowhere Differentiability

Theorem 1. *With probability one, the map $t \mapsto B_t$ is nowhere differentiable on $(0, \infty)$. Moreover, the right derivative at 0 does not exist.*

Proof. Fix $T > 0$. For $n \in \mathbb{N}$, set $h_n = 2^{-n}$ and

$$\Delta_{k,n} = B_{(k+1)h_n} - B_{kh_n},$$

for integers k such that the corresponding interval is contained in $[0, T+1]$. The random variables $\Delta_{k,n}$ are independent and distributed as $N(0, h_n)$.

For $M \in \mathbb{N}$, let $A_{n,M}$ be the event that there exists an integer k satisfying

$$1 \leq k, \qquad kh_n \leq T, \qquad (k+2)h_n \leq T+1,$$

such that

$$|\Delta_{k-1,n}|, \quad |\Delta_{k,n}|, \quad |\Delta_{k+1,n}| \leq 4Mh_n.$$

Since the density of a standard normal random variable is bounded by $(2\pi)^{-1/2}$,

$$\mathbb{P}(|N(0,1)| \leq x) \leq \sqrt{\frac{2}{\pi}}\, x, \qquad x \geq 0.$$

Hence, for a fixed triple of consecutive increments,

$$\mathbb{P}(|\Delta_{k-1,n}|, |\Delta_{k,n}|, |\Delta_{k+1,n}| \leq 4Mh_n) \leq \left(CM\sqrt{h_n}\right)^3$$

for an absolute constant $C > 0$. There are at most $C_T h_n^{-1}$ possible values of k, so by the union bound,

$$\mathbb{P}(A_{n,M}) \leq C_T M^3 h_n^{1/2} = C_T M^3 2^{-n/2}.$$

Therefore

$$\sum_{n=1}^{\infty} \mathbb{P}(A_{n,M}) < \infty.$$

By the first Borel–Cantelli lemma, for each fixed M, almost surely only finitely many of the events $A_{n,M}$ occur.

Now suppose that a sample path is differentiable at some $t \in (0, T)$. Then there are $M \in \mathbb{N}$ and $\delta > 0$ such that

$$|B_s - B_t| \leq M|s - t| \qquad \text{whenever } |s - t| < \delta.$$

For all sufficiently large n, choose k with

$$kh_n \leq t < (k+1)h_n.$$

The four grid points $(k-1)h_n, kh_n, (k+1)h_n, (k+2)h_n$ then lie within distance $2h_n < \delta$ of t. Consequently, for $j = k-1, k, k+1$,

$$\begin{aligned} |\Delta_{j,n}| &\le \left|B_{(j+1)h_n} - B_t\right| + |B_{jh_n} - B_t| \\ &\le 4Mh_n. \end{aligned}$$

Thus $A_{n,M}$ occurs for every sufficiently large n, contradicting Borel–Cantelli. Taking the countable intersection over $M \in \mathbb{N}$ shows that almost surely there is no differentiability point in $(0, T)$.

Applying this to every positive integer T gives nowhere differentiability on $(0, \infty)$ almost surely.

It remains to consider $t = 0$. If the right derivative at 0 existed, then for some $M \in \mathbb{N}$,

$$|B_h| \le Mh$$

for all sufficiently small $h > 0$. In particular, with $h_n = 2^{-n}$,

$$\mathbb{P}(|B_{h_n}| \le Mh_n) = \mathbb{P}\Big(|N(0,1)| \le M\sqrt{h_n}\Big) \le CM2^{-n/2}.$$

The sum of these probabilities is finite, so Borel–Cantelli implies that almost surely this event occurs only finitely often. Hence the right derivative at 0 cannot exist. $\square$

2 Infinite Total Variation

For a function $f : [a, b] \to \mathbb{R}$, its total variation on $[a, b]$ is

$$V_a^b(f) = \sup_{\Pi} \sum_{i=1}^{m} |f(t_i) - f(t_{i-1})| \,,$$

where the supremum is over all partitions $\Pi = \{a = t_0 < t_1 < \cdots < t_m = b\}$.

Lemma 2. *For every fixed* $0 \le a < b$*, along the uniform dyadic partitions*

$$t_{k,n} = a + k\frac{b-a}{2^n}, \qquad k = 0, \dots, 2^n,$$

we have

$$\sum_{k=1}^{2^n} \big(B_{t_{k,n}} - B_{t_{k-1,n}}\big)^2 \longrightarrow b - a$$

almost surely.

Proof. Set

$$Q_n = \sum_{k=1}^{2^n} \big(B_{t_{k,n}} - B_{t_{k-1,n}}\big)^2 .$$

The increments are independent $N(0, (b-a)2^{-n})$ random variables. Therefore

$$\mathbb{E}[Q_n] = b - a.$$

For $X \sim N(0, \sigma^2)$, one has $\mathrm{Var}(X^2) = 2\sigma^4$. Hence

$$\mathrm{Var}(Q_n) = 2^n \cdot 2\left(\frac{b-a}{2^n}\right)^2 = 2(b-a)^2 2^{-n}.$$

By Chebyshev's inequality, for every $\varepsilon > 0$,

$$\mathbb{P}(|Q_n - (b-a)| > \varepsilon) \le \frac{2(b-a)^2}{\varepsilon^2} 2^{-n}.$$

The right-hand side is summable in n. Thus Borel–Cantelli yields $Q_n \to b - a$ almost surely. $\square$

Theorem 3. *With probability one, the Brownian sample path has infinite total variation on every nontrivial interval* $[a,b] \subset [0,\infty)$.

Proof. First restrict to rational endpoints $a, b \in \mathbb{Q}$ with $0 \le a < b$. There are only countably many such intervals, so by Lemma 2, with probability one, simultaneously for every rational $a < b$,

$$Q_n(a,b) \longrightarrow b - a > 0.$$

Fix a sample path in this probability-one event and suppose, for contradiction, that its total variation on some rational interval $[a,b]$ is finite. Brownian paths are continuous, hence uniformly continuous on $[a,b]$. Along the dyadic partitions, therefore,

$$\max_{1\le k\le 2^n} \left|B_{t_{k,n}} - B_{t_{k-1,n}}\right| \longrightarrow 0.$$

Moreover,

$$\begin{aligned} Q_n(a,b) &= \sum_{k=1}^{2^n} \left|B_{t_{k,n}} - B_{t_{k-1,n}}\right|^2 \\ &\le \left(\max_{1\le k\le 2^n} \left|B_{t_{k,n}} - B_{t_{k-1,n}}\right|\right) V_a^b(B). \end{aligned}$$

If $V_a^b(B) < \infty$, the right-hand side tends to 0, contradicting $Q_n(a,b) \to b - a > 0$. Thus the variation is infinite on every rational interval.

Finally, every nontrivial interval $[u,v]$ contains a rational subinterval $[a,b]$ with $u < a < b < v$. Since total variation is monotone under restriction,

$$V_u^v(B) \ge V_a^b(B) = \infty.$$

Hence almost surely the Brownian path has infinite total variation on every nontrivial interval. □

3 Conclusion

Corollary 4. *There exists an event of probability one on which, simultaneously,*

(1) $t \mapsto B_t$ *is nowhere differentiable on* $[0,\infty)$*, with differentiability at* 0 *interpreted as right differentiability;*

(2) $t \mapsto B_t$ *has infinite total variation on every nontrivial interval.*

Proof. Intersect the two probability-one events supplied by Theorems 1 and 3. □