Symmetry and Structure in Tensor Algebra

A Student of Linear Algebra

October 7, 2025

Abstract

This article explores the rich internal structure of tensor spaces by examining symmetry properties. We introduce the fundamental concepts of symmetric and antisymmetric tensors, leading to the construction of two crucial algebraic structures: the symmetric algebra and the exterior algebra. The latter, with its associated wedge product, forms the foundation for the theory of differential forms. By developing this purely algebraic framework first, we set the stage for a deeper understanding of the metric tensor as an essential tool that endows this algebra with geometric meaning.

1 Introduction: Decomposing Tensor Spaces

We have established that the set of all tensors over a vector space $\mathcal{V}$ forms a graded algebra under the outer product. However, the spaces $T^p_q(\mathcal{V})$ themselves possess a great deal of internal structure. A powerful way to analyze this structure is to study how tensors behave under the permutation of their arguments. This leads to a natural decomposition of tensor spaces into subspaces of tensors with specific symmetries. For simplicity, we will focus our attention on purely covariant tensors in $T^0_k(\mathcal{V})$, as the generalization is straightforward.

2 Symmetric and Antisymmetric Tensors

Let S_k be the symmetric group of permutations of $\{1, 2, \dots, k\}$. For any permutation $\sigma \in S_k$, we can define its action on a tensor $T \in T^0_k(\mathcal{V})$ by permuting its vector arguments.

Definition 2.1. *Let $T \in T^0_k(\mathcal{V})$ and $\sigma \in S_k$. The action of σ on T produces a new tensor $\sigma T \in T^0_k(\mathcal{V})$ defined by:*

$$(\sigma T)(v_1, \dots, v_k) := T(v_{\sigma(1)}, \dots, v_{\sigma(k)})$$

This action allows us to define the key symmetry classes.

Definition 2.2. *Let $T \in T^0_k(\mathcal{V})$.*

1. *T is **symmetric** if $\sigma T = T$ for all $\sigma \in S_k$. This is equivalent to saying that T is unchanged by the swapping of any two arguments:*

$$T(\dots, u, \dots, v, \dots) = T(\dots, v, \dots, u, \dots)$$

2. *T is **antisymmetric** (or **alternating**) if $\sigma T = \operatorname{sgn}(\sigma) T$ for all $\sigma \in S_k$, where $\operatorname{sgn}(\sigma)$ is the sign of the permutation. This is equivalent to saying that T changes sign upon the swapping of any two arguments:*

$$T(\dots, u, \dots, v, \dots) = -T(\dots, v, \dots, u, \dots)$$

The space of symmetric k-tensors is denoted $S^k(\mathcal{V}^*)$, and the space of alternating k-tensors is denoted $\Lambda^k(\mathcal{V}^*)$.

Remark 2.1. *A direct consequence of the definition is that if any two arguments of an alternating tensor are identical, the result is zero: $T(\dots, v, \dots, v, \dots) = 0$.*

2.1 The Symmetrization and Alternation Operators

We can create (anti)symmetric tensors from arbitrary ones using projection operators.

Definition 2.3. *The* ***symmetrization operator*** $\mathrm{Sym} : T_k^0(\mathcal{V}) \to S^k(\mathcal{V}^*)$ *and the* ***alternation operator*** $\mathrm{Alt} : T_k^0(\mathcal{V}) \to \Lambda^k(\mathcal{V}^*)$ *are defined as:*

$$\mathrm{Sym}(T) := \frac{1}{k!} \sum_{\sigma \in S_k} \sigma T$$
$$\mathrm{Alt}(T) := \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \sigma T$$

These operators act as projections onto the respective subspaces.

In Components. If a tensor T has components $T_{i_1 \ldots i_k}$, then:

- T is symmetric if $T_{i_{\sigma(1)} \ldots i_{\sigma(k)}} = T_{i_1 \ldots i_k}$ for any permutation σ.
- T is antisymmetric if $T_{i_{\sigma(1)} \ldots i_{\sigma(k)}} = \mathrm{sgn}(\sigma) T_{i_1 \ldots i_k}$. For example, $A_{ij} = -A_{ji}$.

The components of the symmetrized tensor are often denoted with parentheses, $T_{(i_1 \ldots i_k)}$, and the alternated tensor with square brackets, $T_{[i_1 \ldots i_k]}$.

3 The Exterior Algebra and the Wedge Product

The space of alternating tensors, $\Lambda^k(\mathcal{V}^*)$, also known as the space of k**-covectors** or k**-forms**, possesses a particularly beautiful and important algebraic structure. We can define a new product operation that preserves the property of alternation.

Definition 3.1 (Wedge Product)**.** *Let $\alpha \in \Lambda^k(\mathcal{V}^*)$ and $\beta \in \Lambda^l(\mathcal{V}^*)$. Their* ***wedge product*** *(or exterior product), denoted $\alpha \wedge \beta$, is an element of $\Lambda^{k+l}(\mathcal{V}^*)$ defined as:*

$$\alpha \wedge \beta := \frac{(k+l)!}{k! l!} \mathrm{Alt}(\alpha \otimes \beta)$$

The normalization factor is chosen to simplify formulas for basis vectors. The wedge product has the following fundamental properties:

1. **Bilinearity**: $(c\alpha_1 + \alpha_2) \wedge \beta = c(\alpha_1 \wedge \beta) + (\alpha_2 \wedge \beta)$.
2. **Associativity**: $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$.
3. **Graded Anticommutativity**: $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$.

This last property is crucial. It implies that the wedge product of a k-form with itself is zero if k is odd.

The direct sum of these spaces, $\Lambda(\mathcal{V}^*) = \bigoplus_{k=0}^{\dim(\mathcal{V})} \Lambda^k(\mathcal{V}^*)$, equipped with the wedge product, forms a graded algebra called the **exterior algebra** of $\mathcal{V}^*$.

3.1 The Wedge Product in Components

While the abstract definition is powerful, the component form is highly intuitive. Let $\{\epsilon^i\}$ be a basis for $\mathcal{V}^*$.

Example 3.1. *Let $\alpha = \alpha_i \epsilon^i$ and $\beta = \beta_j \epsilon^j$ be two 1-forms. Their wedge product is:*

$$\begin{aligned}\alpha \wedge \beta &= (\alpha_i \epsilon^i) \wedge (\beta_j \epsilon^j) = \alpha_i \beta_j (\epsilon^i \wedge \epsilon^j) \\ &= \frac{1}{2} \alpha_i \beta_j (\epsilon^i \otimes \epsilon^j - \epsilon^j \otimes \epsilon^i) \\ &= \sum_{i<j} (\alpha_i \beta_j - \alpha_j \beta_i)(\epsilon^i \wedge \epsilon^j)\end{aligned}$$

The components of this 2-form are $(\alpha \wedge \beta)_{ij} = \alpha_i \beta_j - \alpha_j \beta_i$.

A basis for $\Lambda^k(\mathcal{V}^*)$ is given by the set $\{\epsilon^{i_1} \wedge \cdots \wedge \epsilon^{i_k}\}$ for all ordered multi-indices $1 \le i_1 < i_2 < \cdots < i_k \le \dim(\mathcal{V})$. This implies that $\dim(\Lambda^k(\mathcal{V}^*)) = \binom{\dim(\mathcal{V})}{k}$.

4 Revisiting the Metric: From Algebra to Geometry

Up to this point, our entire discussion of symmetry and the exterior algebra has been purely algebraic, requiring no additional structure on $\mathcal{V}$. This is where the metric tensor enters, not as an algebraic necessity, but as a **geometric tool**. A metric tensor g (an inner product) on $\mathcal{V}$ allows us to answer geometric questions.

Motivation:

- How long is a vector? We need a metric: $\|v\|^2 = g(v, v)$.
- What is the angle between two vectors? We need a metric: $\cos\theta = g(u, v)/(\|u\|\|v\|)$.
- What is the "volume" of a parallelepiped spanned by k vectors? This requires a metric.
- Is there a natural way to relate the space of k-forms $\Lambda^k(\mathcal{V}^*)$ to the space of $(n-k)$-forms $\Lambda^{n-k}(\mathcal{V}^*)$?

The last two questions are answered by using the metric to enrich the exterior algebra. A metric g on $\mathcal{V}$ induces a canonical inner product on each space $\Lambda^k(\mathcal{V}^*)$. This allows us to define orthonormal bases of k-forms and, crucially, to define a unique, metric-dependent unit n-form called the **volume form**, which corresponds to our intuitive notion of volume.

The existence of an inner product and a volume form then allows for the definition of the **Hodge star operator** ($\star$), a canonical isomorphism $\star : \Lambda^k(\mathcal{V}^*) \to \Lambda^{n-k}(\mathcal{V}^*)$. This operator is fundamental in geometry and physics (e.g., in Maxwell's equations) and it *cannot be defined without a metric.*

5 Conclusion

The space of tensors is not monolithic; it possesses a rich internal structure governed by symmetries. By isolating the symmetric and, more importantly, the antisymmetric tensors, we uncover the exterior algebra—a powerful and elegant framework with its own unique product rule ($\wedge$). This entire structure is built on the axioms of a vector space alone.

The metric tensor should now be seen in a new light. It is not just another type-(0,2) tensor. It is a special tensor that we *introduce* to our vector space to endow it with a geometric character. It serves as the bridge between the abstract, algebraic world of forms and the concrete, geometric world of lengths, angles, volumes, and dualities. The subsequent study of operators like the Hodge star will make this role explicit and demonstrate that much of what we consider "geometry" is precisely the interplay between the exterior algebra and a metric tensor.