A Natural Proof of the Dominated Convergence Theorem

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The Dominated Convergence Theorem is a grand result in the area of Lebesgue integration. It deals with the problem of interchanging limits and integrals, i.e., whether

$$\lim_{n\to\infty}\int f_n = \int f$$

for $f_n \to f$. The theorem demonstrates that Lebesgue integration indeed fixes a serious deficiency of Riemann integration, at least to a reasonably satisfactory extent.

Theorem (Dominated Convergence Theorem, DCT)**.** *Suppose $(X, \mathcal{S}, \mu)$ is a measure space, $f : X \to [-\infty, \infty]$ is measurable, and $\{f_n\}_{n\in\mathbf{N}}$ is a sequence of measurable functions such that $f_n \to f$ a.e. If there exists a nonnegative measurable function $g \in \mathcal{L}^1(X, \mathcal{S}, \mu)$ s.t.*

$$|f_n(x)| \leq g(x)$$

for every $n \in \mathbf{N}$ and almost every $x \in X$, then

$$\lim_{n\to\infty}\int f_n \,\mathrm{d}\mu = \int f \,\mathrm{d}\mu.$$

The traditional proof of DCT uses Fatou's Lemma and is sketched in 1.2. However, the proof is clean but not natural, at least to me. This article presents an alternative proof using Egorov's Theorem, which is slightly more complicated but quite intuitive.

1 Background

For the rest of the article, suppose $(X, \mathcal{S}, \mu)$ is a measure space.

1.1 A deficiency of Riemann integration

One problem with Riemann integration is that it does not work well with limits. Integrals of (pointwise) limits do not necessarily equal limits of integrals; limits of Riemann integrable functions are not necessarily Riemann integrable[1]. Below are two interesting examples.

Example. Define $f_n : [0, 1] \to \mathbf{R}$ for each $n \in \mathbf{N}$ by

$$f_n(x) = \lim_{m\to\infty} |\cos(2n!\pi x)|^m .$$

Then $\int f_n = 0$ for each n but

$$f(x) = \lim_{n\to\infty} f_n(x) = \chi_{\mathbf{Q}}$$

is the Dirichlet function and is not Riemann integrable.

[1] In contrast, the pointwise limit of measurable functions is guaranteed to be measurable.

Example (Example 1.16 of *Measure, Integration, and Real Analysis* by Sheldon Axler)**.** Suppose $\{r_n\}_{n\in\mathbf{N}}$ is an enumeration of $(0,1)\cap\mathbf{Q}$. Define $f_n:[0,1]\to\mathbf{R}$ for each $n\in\mathbf{N}$ by

$$f_n(x)=\begin{cases}\frac{1}{\sqrt{x-r_n}} & \text{if } x>r_n,\\ 0 & \text{otherwise.}\end{cases}$$

Define $f:[0,1]\to[0,\infty]$ by

$$f(x)=\sum_{n=0}^{\infty}\frac{f_n(x)}{2^n}.$$

Because $\mathbf{Q}$ is dense in $\mathbf{R}$, f is unbounded on every nontrivial interval. Thus f is not Riemann integrable on every nontrivial subinterval of $[0,1]$. However, we should expect the area under the graph of f to be less than 4 rather than undefined, since the area under the graph of each f_n is less than 2.

Although we have the following positive result, the theorem has two problems: it requires the hypothesis that the limit function is Riemann integrable, and mathematicians have not yet found a clean, elementary, Riemann-integration-based proof of it. In fact, this suggests that Riemann integration is not the ideal theory of integration. See Theorem 1.18 of *Measure, Integration, and Real Analysis* by Sheldon Axler and section 14.2.3 of 《数学分析习题课讲义（第 2 版）（下册)》 by 谢惠民 et al.

Theorem (Arzelà's Bounded Convergence Theorem)**.** *Suppose $\{f_n\}_{n\in\mathbf{N}}$ is a sequence of Riemann integrable functions on $[a,b]$, $\{f_n\}$ is uniformly bounded, and $f_n\to f$. If f is Riemann integrable, then*

$$\lim_{n\to\infty}\int_a^b f_n=\int_a^b f.$$

1.2 Traditional proof of DCT by Fatou's Lemma

We state Fatou's Lemma and sketch the proof of DCT using Fatou's Lemma below.

Theorem (Fatou's Lemma)**.** *Suppose $\{f_n\}_{n\in\mathbf{N}}$ is a sequence of nonnegative measurable functions. Then*

$$\int\left(\liminf_{n\to\infty}f_n\right)\mathrm{d}\mu\le\liminf_{n\to\infty}\int f_n\,\mathrm{d}\mu.$$

Proof sketch of DCT using Fatou's Lemma. Notice that $g-f_n\ge 0$ a.e. and $g+f_n\ge 0$ a.e. By Fatou's Lemma,

$$\int\left(\liminf_{n\to\infty}(g-f_n)\right)\mathrm{d}\mu\le\liminf_{n\to\infty}\int(g-f_n)\,\mathrm{d}\mu\implies\int f\,\mathrm{d}\mu\ge\limsup_{n\to\infty}\int f_n\,\mathrm{d}\mu,$$

$$\int\left(\liminf_{n\to\infty}(g+f_n)\right)\mathrm{d}\mu\le\liminf_{n\to\infty}\int(g+f_n)\,\mathrm{d}\mu\implies\int f\,\mathrm{d}\mu\le\liminf_{n\to\infty}\int f_n\,\mathrm{d}\mu.$$

The proof is completed. □

2 Some Preparation

We first state Egorov's Theorem.

Theorem (Egorov's Theorem)**.** *Suppose $\mu(X)<\infty$, $\{f_n\}_{n\in\mathbf{N}}$ is a sequence of measurable functions, and $f_n\to f$. Then $\forall\varepsilon>0$, $\exists E\in\mathcal{S}$ s.t. $\mu(E^c)<\varepsilon$ and f_n converges uniformly to f on E.*

To use Egorov's Theorem to prove DCT, we still need to address two problems.

(1) $\mu(X)$ might not be finite.

(2) We need to control the integral over the set on which f_n does not converge uniformly.

Lemma 2 will solve the first problem, and Lemma 1 will solve the second.

2.1 Integrals over small sets are small

Lemma 1. *Suppose $f \in \mathcal{L}^1(X, \mathcal{S}, \mu)$ is nonnegative. Then $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. for every $B \in \mathcal{S}$ with $\mu(B) < \delta$,*

$$\int_B f \, d\mu < \varepsilon.$$

Proof. Suppose $\varepsilon > 0$. The desired result is trivial for simple functions. For the general case, take a simple measurable function h s.t. $0 \le h \le f$ and $\int (f - h) \, d\mu < \varepsilon/2$. Then $\exists \delta > 0$ s.t. for every $B \in \mathcal{S}$ with $\mu(B) < \delta$,

$$\int_B h \, d\mu < \frac{\varepsilon}{2}.$$

Then

$$\int_B f \, d\mu \le \int (f - h) \, d\mu + \int_B h \, d\mu < \varepsilon$$

for every $B \in \mathcal{S}$ with $\mu(B) < \delta$, as desired. $\square$

2.2 Mass of integrable functions concentrates on sets of finite measure

Lemma 2. *Suppose $f \in \mathcal{L}^1(X, \mathcal{S}, \mu)$ is nonnegative. Then $\forall \varepsilon > 0$, $\exists E \in \mathcal{S}$ s.t. $\mu(E) < \infty$ and*

$$\int_{E^c} f \, d\mu < \varepsilon.$$

Proof. Suppose $\varepsilon > 0$. The desired result is trivial for simple functions. For the general case, take a simple measurable function h s.t. $0 \le h \le f$ and $\int (f - h) \, d\mu < \varepsilon/2$. Then $\exists E \in \mathcal{S}$ s.t. $\mu(E) < \infty$ and

$$\int_{E^c} h \, d\mu < \frac{\varepsilon}{2}.$$

Then

$$\int_{E^c} f \, d\mu \le \int (f - h) \, d\mu + \int_{E^c} h \, d\mu < \varepsilon,$$

as desired. $\square$

3 A Natural Proof of DCT by Egorov's Theorem

Fix $\varepsilon > 0$. By Lemma 2, let $X_0 \in \mathcal{S}$ be s.t. $\mu(X_0) < \infty$ and

$$\int_{X_0^c} g \, d\mu < \frac{\varepsilon}{5}.$$

By Lemma 1, let $\delta > 0$ be s.t for every $B \in \mathcal{S}$ with $\mu(B) < \delta$,

$$\int_B g \, d\mu < \frac{\varepsilon}{5}.$$

By Egorov's Theorem, $\exists E \subseteq X_0$ s.t. $E \in \mathcal{S}$, $\mu(X_0 \setminus E) < \delta$, and f_n converges uniformly to f on E. Then

$$\begin{aligned}
\left|\int (f_n - f)\,\mathrm{d}\mu\right| &\le \int |f_n - f|\,\mathrm{d}\mu \\
&\le \int_E |f_n - f|\,\mathrm{d}\mu + \int_{X_0 \setminus E} (|f_n| + |f|)\,\mathrm{d}\mu + \int_{X_0^c} (|f_n| + |f|)\,\mathrm{d}\mu \\
&\le \int_E |f_n - f|\,\mathrm{d}\mu + 2\int_{X_0 \setminus E} g\,\mathrm{d}\mu + 2\int_{X_0^c} g\,\mathrm{d}\mu \\
(n \ge N) \quad &\le \varepsilon,
\end{aligned}$$

where $N \in \mathbf{N}$ is s.t.

$$|f_n(x) - f(x)| < \frac{\varepsilon}{5\mu(E)}$$

for every $n \ge N$ and every $x \in E$. The proof is completed. □